

Spherically symmetric dark energy structures in a generalized Chaplygin gas model

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Abstract

Spherically symmetric dark energy structures are investigated in the framework of a generalized Chaplygin gas (GCG), which has an equation of state of the form $P = -A/\rho^\alpha$. We also study these in a modified GCG equation of state, which includes a matter term, i.e. $P = \sigma^2 \rho - A/\rho^\alpha$. The results of the latter are then compared with some observational data on low-surface-brightness galaxies which are supposed to be dominated by dark matter.

1. Introduction

The generalized Chaplygin gas is one of the promising candidates to explain the present accelerated expansion of the universe. It is a simple model unifying dark energy and dark matter based on a perfect fluid having negative pressure [1]. The equation of state of the generalized Chaplygin gas is given by

$$P_{ch} = -\frac{A}{\rho_{ch}^\alpha} , \quad (1)$$

where A and α are positive constants and $0 \leq \alpha \leq 1$ and P_{ch} and ρ_{ch} are the generalized Chaplygin gas pressure and density, respectively. The energy conservation in the context of FRW cosmology yields an expression for the density ρ in terms of the scale factor a as

$$\rho_{ch} = \left(A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}} . \quad (2)$$

The corresponding expression for the original Chaplygin gas is the special case $\alpha = 1$ of this equation. The generalized Chaplygin gas model, like the original Chaplygin gas model, evolves first as dust and then as cosmological constant at late times. However, in its intermediate stages, it behaves as a mixture of a cosmological constant and a perfect fluid with an equation of state $P = \lambda\rho$. The parameter λ in the generalized Chaplygin equation makes this model more flexible for comparison against observations. In this paper, we will investigate the possible existence of spherically symmetric dark energy structures in the framework of the generalized Chaplygin Gas (GCG).

2. Structure equations and solutions

Starting with a line element

$$ds^2 = e^\alpha dt^2 - e^\beta dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (3)$$

and choosing

$$e^\beta = \frac{1}{1 - \frac{2GM}{r}} \quad (4)$$

and expressing the density in terms of the enclosed mass, the conservation equation $T^\nu_{\mu;\nu} = 0$ yields

$$P' + \frac{\alpha'}{2} (\rho + P) = 0 \quad (5)$$

and therefore

$$\frac{P'}{P + \rho} + \frac{G \left(\frac{M}{r^2} + 4\pi r P \right)}{1 - \frac{2GM}{r}} = 0 . \quad (6)$$

Eq.(6) is a generalization of the equilibrium Euler-Poisson equation. Indeed, for small values of P/ρ , $r^3 P$, and $2GM/r$, it is possible to obtain an approximation of eq.(6) as

$$r^2 \frac{P'}{\rho} + GM = 0 . \quad (7)$$

Finally, combining eq.(6) with the equation

$$M' = 4\pi r^2 \rho , \quad (8)$$

we can get

$$\frac{1}{r^2} \left(r^2 \frac{P'}{\rho} \right)' + 4\pi G \rho = 0 . \quad (9)$$

This is a useful approximate equation to find a scaling solution for any arbitrary equation of state.

In order to determine the scaling solution for the generalized Chaplygin gas model density, let us assume that the density profile, we are looking for, is

$$\rho(r) = \frac{C}{(H_0 r)^\gamma} . \quad (10)$$

Here H_0 is the present day Hubble constant which corresponds to the critical density ρ_c , as $H_0^2 = 8\pi G\rho_c/3$, C and γ being constants to be determined. Combining equations (9) and (10) yields

$$\gamma \left[1 + \gamma(1 + \alpha) \right] \alpha A r^{\gamma(1+\alpha)-2} H_0^{\gamma(2+\alpha)} = 4\pi G r^{-\gamma} C^{2+\alpha} . \quad (11)$$

Matching the powers and the coefficients of both terms on left-hand side of this equation, the values of γ and C are found to be

$$\gamma = \frac{2}{2 + \alpha} \quad (12)$$

and

$$C = \left[\frac{4\alpha(4 + 3\alpha)}{3(2 + \alpha)^2} A \rho_c \right]^{\frac{1}{2+\alpha}} , \quad (13)$$

respectively. To fix A , let us consider a generalized Chaplygin gas dominated by the FRW model. According to this model the square of the Hubble parameter is given as

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho = \frac{8\pi G}{3} \left(A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}} . \quad (14)$$

The present day critical density is

$$\rho_c = (A + B)^{\frac{1}{1+\alpha}} , \quad (15)$$

choosing $a = 1$ at present. The constant B can also be expressed in terms of the matter fraction Ω at high redshift ($a \ll 1$) as $B = (\Omega \rho_c)^{1+\alpha}$. The constant A is then found to be

$$A = \rho_c^{1+\alpha} (1 - \Omega^{1+\alpha}) \quad (16)$$

and therefore,

$$\rho(r) = \frac{\rho_c \overline{C}}{(H_0 r)^{2/(2+\alpha)}} , \quad (17)$$

with

$$\overline{C} = \left[\frac{4\alpha(4+3\alpha)}{3(2+\alpha)^2} (1 - \Omega^{1+\alpha}) \right]^{\frac{1}{2+\alpha}} . \quad (18)$$

Based on the assumptions in order to get the approximate equation (9), this solution is subject to the condition

$$\left| \frac{P(r)}{\rho(r)} \right| = \frac{A}{[\rho(r)]^{1+\alpha}} = (1 - \Omega^{1+\alpha})^{\frac{1}{2+\alpha}} \left[\left(\frac{3(2+\alpha)^2}{4\alpha(4+3\alpha)} \right) H_0^2 r^2 \right]^{\frac{1+\alpha}{2+\alpha}} \ll 1 \quad (19)$$

or essentially $r \ll H_0^{-1}$. As can be seen from eq.(17), the spherically symmetric dark energy density can be as large as $10^3 \rho_c$, on a scale of 100 kpc. The metric inside the spherically symmetric Chaplygin gas is the Schwarzschild metric, while the background is a FRW metric. The spherically symmetric dark energy is embedded in the background fluid. In order to fit this spherically symmetric dark energy properly into the background, the pressure in both sides of the boundary must be equal [2-3], i.e.

$$P_{FRW} = P_{SCW} , \quad (20)$$

which implies that the densities in the two regions must also be equal, as we are dealing with the same Chaplygin gas equation of state both inside and outside the spherically symmetric dark energy, i.e.

$$\rho_{FRW} = \rho_{SCW} . \quad (21)$$

The density profile inside the spherically symmetric system, with Schwarzschild metric, is $M' = 4\pi r^2 \rho$. The outside fluid density at high redshift ($a \ll 1$) is

then given as

$$\rho_{FRW} = \frac{\rho_c \Omega}{a^3} \quad \text{for } a \ll 1 . \quad (22)$$

Plugging the corresponding densities into eq.(26), we obtain

$$\frac{\rho_c \bar{C}}{(H_0 r)^{\frac{2}{2+\alpha}}} = \frac{\rho_c \Omega}{a^3} . \quad (23)$$

Solving for the radius r in terms of the scale factor a and the constant B , we arrive at

$$r(t) = \tilde{B} a^{3+\frac{3\alpha}{2}} , \quad (24)$$

with the constant

$$\tilde{B} = H_0^{-1} \left[\frac{\bar{C}}{\Omega} \right]^{1+\frac{\alpha}{2}} . \quad (25)$$

3. Discussion and Results

In their recent paper [4], Bertolami and Paramos studied a spherically symmetric dark energy structure (they called “dark energy”), using a polytropic equation of state of negative index. They argued that there are conditions for such objects to form, due to density fluctuations in the background of the generalized Chaplygin gas. According to these authors, the condition that helps the initial fluctuation to grow, is that the sound velocity at the surface of the dark star must be less than the initial expansion velocity of the dark star.

According to our solution, the expansion rate is calculated to be

$$\dot{r} = (3 + 3\alpha/2) \left[\frac{\bar{C}^{2+\alpha}}{\Omega^{1+\alpha}} \right]^{1/2} a^{3(1+\alpha)/2} \quad (26)$$

and the speed of sound is given by

$$c_s^2 = \frac{\alpha A}{\rho^{1+\alpha}} = -\alpha \frac{P}{\rho} \quad (27)$$

$$c_s = \left[\frac{\alpha \rho_c^{1+\alpha} (1 - \Omega^{1+\alpha})}{\left[\frac{\rho_c \Omega}{\alpha^3} \right]^{1+\alpha}} \right]^{1/2} = \left[\frac{\alpha (1 - \Omega^{1+\alpha})}{\Omega^{1+\alpha}} \right]^{1/2} a^{3(1+\alpha)/2} . \quad (28)$$

The ratio of the expansion rate to the speed of sound is

$$\frac{\dot{r}}{c_s} = \frac{3}{2} (2 + \alpha) \frac{2\sqrt{4 + 3\alpha}}{\sqrt{3}(2 + \alpha)} = \sqrt{12 + 9\alpha} > 1 . \quad (29)$$

Then it is true that the condition of Bertolami and Paramos [4] holds. However, this is not sufficient to conclude that such objects exist. An additional point that has to be checked, is the size of the initial density contrast. The average density is

$$\frac{m(r)}{\frac{4}{3} \pi r^3} = \frac{3(2 + \alpha)}{4 + 3\alpha} \bar{\rho}(\alpha) , \quad (30)$$

which means that the density contrast is

$$\delta = \frac{2}{4 + 3\alpha} . \quad (31)$$

Thus this density contrast is independent of time. For particular case $\alpha = 1$, the density contrast is about 0.28, which is much larger than the value 10^{-5} predicted by CMB observations. Therefore, even if the expansion rate of the dark star is larger than the speed of sound, the density contrast is not sufficient to grow and form such structures.

Generally, a non-vanishing speed of sound is a major problem for structure formation in all unified models of dark energy and dark matter. Such systems have a characteristic scale, the sonic horizon, below which the pressure frustrates gravitational structure formation. Small perturbations of scales below this characteristic scale die off without any noticeable effect. Combining the continuity and the Euler-Poisson equations for the case of vanishing shear

and rotation, subtracting the background, and finally changing the variable from t to a , one gets [5]

$$\alpha^2 \delta'' + \frac{3}{2} \alpha \delta' - \frac{3}{2} \delta (1 + \delta) - \frac{4}{3} \frac{(\alpha \delta')^2}{1 + \delta} - \frac{1 + \delta}{a^2 H^2} \frac{\partial}{\partial x_i} \left(\frac{c_s^2}{1 + \delta} \frac{\partial \delta}{\partial x_i} \right) = 0 , \quad (32)$$

where $\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$ is the density contrast, c_s is the speed of sound, given by

$$c_s^2 = \frac{dP}{d\rho} = \frac{\alpha A}{\rho^\alpha} \quad (33)$$

for the Chaplygin gas model.

The linear solution for the perturbative density contrast of eq.(19), well discussed by Fabris *et al.* [29], is given as

$$\delta_{per}(k, \alpha) \propto \alpha^{-1/4} J_{5/14}(d_s k) , \quad (34)$$

where $J_\nu(z)$ is the Bessel function and k is the comoving wave number and d_s is the sonic horizon. This means that $\delta_{per} \sim a$ for $d_s k \ll 1$, and it oscillates with a decaying amplitude when $d_s k \gg 1$. This is because the non-zero speed of sound, and hence non-zero pressure, opposes the effect of gravity below that characteristic (sonic horizon) scale. This characteristic scale is, for the original Chaplygin gas model, given by [5]

$$d_s = \int_0^a \frac{c_s d\alpha}{a^2 H} = \frac{2}{7} \frac{(1 - \Omega^2)^{1/2}}{\Omega^{3/2}} \frac{a^{7/2}}{H_0} , \quad (35)$$

where $\bar{c}_s = \sqrt{A/\bar{\rho}}$ and $\Omega = \sqrt{B/(A+B)} = \sqrt{B}/\rho_{cr}$. The sonic horizon has a radius of about 0.18 Mpc at a redshift $z \sim 20$. This distance is much larger than the size of the largest galaxy ever observed.

It is necessary to investigate eq.(32) beyond the linear regime, in order to

decide whether initial perturbations can grow to the extent that they are powerful enough to give rise to gravitational condensation or not. Generally, in the nonlinear region there is a possibility, though very small, that initial perturbations can grow unlimited. Bilić, Lindebaum, Tupper and Viollier concluded in their paper [5] that unlike the linear theory, where δ_R finally stops growing by acoustic horizon for any value of R , the perturbation $\delta_R(a)$ grows in the nonlinear regime to infinity at a finite a , for an initial $\delta_R(a_{dec})$ beyond a certain limit, like the case in the dust model.

However, they also showed in the same paper that the fraction of Chaplygin gas which condensates, due to this infinitely large perturbation, is only 1% [5]. This fraction is too small, in order to conclude that the Chaplygin gas favours structure formation. Being a Chaplygin gas system with a non-zero speed of sound ($c_s^2 = dP/d\rho = \alpha A/\rho^2 \neq 0$), the spherically symmetric dark energy that we are discussing is governed by the general perturbation principle of Chaplygin gas mentioned above. This implies that structure formation is impossible, as the effect of gravity is opposed by the pressure. Thus structure formation contradicts the existence of spherically symmetric dark energy structures (dark stars) in the first place.

In addition to the density contrast problem, the rotational velocity profile of the dark energy structures is not consistent with observation. Starting from the density profile determined in eq.(17), the enclosed mass can be calculated as

$$m(r) = \frac{4\pi(2+\alpha)}{4+3\alpha} \rho_c \overline{C} H_0^{-\gamma} r^{\frac{4+3\alpha}{2+\alpha}} . \quad (36)$$

The resulting rotational speed inside the spherically symmetrical dark energy

(it could be a halo of a galaxy or even a certain spherical patch inside a galaxy), is then given by

$$V_c = \left[\frac{3}{2} \frac{2 + \alpha}{4 + 3\alpha} \overline{C} \right]^{1/2} (H_0 r)^{\frac{1+\alpha}{2+\alpha}} . \quad (37)$$

For the special case $\alpha = 1$, the rotational velocity is proportional to $r^{2/3}$. This relation completely contradicts the flat rotational curve observed for galaxies.

However, it is possible to fix this problem by modifying the pressure and the density relation [6]. This can be done by combining a Chaplygin gas pressure with a pressure that depends on density linearly, i.e.

$$P = \sigma^2 \rho - \frac{A}{\rho^\alpha} . \quad (38)$$

According to this modified pressure, the body would have a flat rotational curve at high density, because at such densities, only the first term is dominant. The problem arises when the density is too low. In that case, the second term, the generalized Chaplygin gas term, is dominant and we are faced with the same non-flat rotational curve problem. It is, however, possible to determine the maximum distance or crossover point to which the first term dominates. This point is where the pressures are equal. $P = \sigma^2 \rho$ and that of the generalized Chaplygin gas $P = -A/\rho^\alpha$ with

$$\rho(r) = \frac{\rho_c \overline{C}}{(H_0 r)^{2/(2+\alpha)}} \quad (39)$$

and

$$\overline{C} = \left[\frac{4\alpha(4 + 3\alpha)}{3(2 + \alpha)^2} (1 - \Omega^{1+\alpha}) \right]^{\frac{1}{2+\alpha}} \quad (40)$$

are equal. The density and the rotational velocity corresponding to the equation of state, $P = \sigma^2 \rho$, of the first part in the right-hand side of eq.(29)

are given respectively as [7]

$$\rho = \frac{\sigma^2}{2\pi G r^2} \quad \text{and} \quad V_c = \sigma \sqrt{2} . \quad (41)$$

The constant σ can be fixed by comparing the second equation with the value of the circular velocity from a real rotational curve.

Plugging in the corresponding values of the densities and the value of the constants A and the constant σ into eq.(24) and finally solving for the crossover radius r_c , we obtain

$$r_c = H_0^{-1} \left[\frac{V_c^4 \overline{C^\alpha}}{3(1 - \Omega^{1+\alpha})} \right]^{\frac{2+\alpha}{4(1+\alpha)}} , \quad (42)$$

where V_c is the rotational velocity, which varies from galaxy to galaxy.

In order to get some numerical insight into the the comparison between the crossover radius and the actual radius of some LSB (low-surface-brightness) and HSB (high-surface-brightness) galaxies, the crossover radius is calculated using the above formula for a Chaplygin gas with $\alpha = 1$ for both LSB and HSB galaxies with their observed rotational velocity. A value of 71 km/s/Mpc and 0.28 are taken for the Hubble constant, H_0 and Ω , respectively.

The real radius, the rotational velocity and the corresponding calculated value of the crossover radius for both LSB and HSB galaxies are given in Table 1 as de Block and McGaugh mentioned in their paper [8].

Name	V_{\max}	$R_{\max}(\text{kpc})$	$R_{\text{cross}}(\text{Mpc})$	Name	V_{\max}	$R_{\max}(\text{kpc})$
F561-1	52	10.1	3.59	DDO154	48	7.6
F563-1	111	17.7	11.18	DDO168	55	3.4
F563V1	30	7.4	1.57	DDO170	66	9.6
F563V2	111	9.2	11.18	N55	87	10.2
F564V4	40	-2	2.42	N257	108	9.9
F565V2	51	8.4	3.48	N300	97	10.6
F567-2	64	11.3	4.90	N801	222	58.7
F568-1	119	14.9	12.42	N1003	115	31.3
F568-3	120	16.5	12.57	N1530	79	8.3
F568V1	124	19	13.21	N2403	136	19.5
F571-8	133	15.6	14.67	N2841	323	81.1
F571V1	73	14.6	5.97	N2903	201	24.2
F571V2	45	3.7	2.89	N2998	214	46.6
F574-1	100	15.4	9.56	N3109	67	8.2
F574-2	40	10.7	2.42	N3198	157	29.9
F577V1	30	8.9	1.57	N5033	222	35.4
F579V1	100	17.3	9.56	N5533	273	74.4
F583-1	85	14.6	7.49	N5585	92	9.6
F583-4	67	10	5.25	N6503	121	22.2
U0128	131	42.3	14.34	N6674	266	64.5
U1230	102	34.7	9.85	N7331	241	36.7
U5005	99	27.8	9.42	U2259	90	7.6
U5750	75	21.8	6.21	U2885	298	72.5
U5999	155	15.3	18.46			

TABLE 1: The table on the left shows the rotational speed, the radius and the corresponding crossover radius for LSB galaxies. Similarly, the table on the right shows the rotational speed, the radius and the corresponding crossover radius for HSB galaxies [8].

As can be seen from Table 1, the real radius of both the LSB and HSB galaxies is of order kpc. However, the corresponding crossover radius for each galaxy is of order Mpc. Thus this modified equation of state works as far as a couple of hundreds Mpc. There is a huge difference between the real radius and the crossover radius. This means that the point at which the modified equation of state fails to hold, is out of the physical extent of these galaxies and hence the modified Chaplygin equation of state can be applied, and it can adequately explain the flat rotational curves. Therefore this equation could be a good alternative to the Chaplygin gas equation, as it could possibly alleviate the rotational curve problem which the Chaplygin gas causes at low densities for spherically symmetric dark energy structure.

It is clear that σ must be a constant that is not characteristic of a particular galaxy. However, the rotational velocity V_c varies from galaxy to galaxy and since σ depends on the rotational velocity as $\sigma = V_c/\sqrt{2}$, σ then differs from galaxy to galaxy. The main problem is generalizing such models as one theory which governs different galaxies.

4. Conclusions

The unrealistic rotational velocity, the very less value of density contrast, together with the small probability (1%) of getting an initial density fluctuation which can result in the formation of large-scale structure exhibited by the spherically symmetrical dark energy structure, leads to the inevitable conclusion that such objects cannot exist under the context of a generalized Chaplygin Gas (GCG) model.

We then modified the GCG equation of state to $P = \sigma^2 \rho - A/\rho^\alpha$. At high density the first term dominates and the resulting rotational curve is consistent with the flat rotational curves observed for galaxies, but at low density the rotational curve is unrealistic. Two types of galaxies (LSB and HSB) are considered to compare the real radius of these sample galaxies with the radius to which the modified equation of state is working for a special case $\alpha = 1$. Fortunately, the point from which this unrealistic rotational curve starts to dominate is completely out of the physical size of the sample galaxies considered. Therefore, this modified equation of state looks capable of explaining the spherically symmetrical dark energy rotational curve. The problem is as the constant σ differs from galaxy to galaxy, there is no general equation, which holds for all galaxies.

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